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The geometry of **M**5-branes and TQFTs

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Abstract

The calculation of the partition function for N **M**5-branes is addressed for the case in which the world-volume wraps a manifold $T^2 \times M_4$, where M_4 is simply connected and Kaehler. This is done in a compactification of **M**-theory which induces the Vafa–Witten theory on M_4 in the limit of vanishing torus volume. The results follow from the equivalence of the BPS spectrum counting in the complementary limit of vanishing M_4 volumes and from a classification of the moduli space of quantum vacua of the supersymmetric twisted theory in terms of associated spectral covers. This reduces the problem of the moduli counting to algebraic equations.

Keywords: **M**5-branes; **M**-theory; World-volume

1. Introduction

The discovery of **D**-branes [1] led string theory to new and unexpected roads. While the low energy effective theory of the strings ending on BPS branes by now seems to be well understood in the flat world-volume case, the problem of formulating and solving it for non-flat branes is still left to be completely settled out. A first major step in this direction has been done in [2], where the naturalness of the twist mechanism for the gauge theory living on the brane was noticed. The root of this stems to the fact that the transverse bosonic degrees of freedom are in fact sections of the normal bundle of the embedded hypersurface on which the branes lay down. On the other hand, the BPS stability condition is guaranteed only if on the brane world-volume some supersymmetry is left [14]. In [2] the supersymmetric cycle condition has been indicated as the root of the twisting procedure.

All this finds a natural explanation once linked to the geometrization of the effective field theory point of view introduced in [3], where the realization of the world-volume embedding equations has been understood as the realization of the SW curve associated to the relevant gauge theory. SW curves [4,5] have been introduced as basic tools to understand

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the geometry of the space of gauge inequivalent vacua in supersymmetric gauge theories. Their RG properties and phases have been therefore better understood in a proper geometric framework. An important development [6] has been given by the understanding that these curves can be seen as spectral curves of an integrable system and that the SW differential generates the RGEs as an explicitly integrable system. On the other hand, these curves constitute the materialization of flat **D**-branes world-volumes in the **M**-theory flat target space. As far as we are concerned, the most important property of this construction is that the curves intrinsic stability has been understood in terms of the stability of the corresponding vacuum in the gauge theory. This means that the moduli space of these curves represent stable gauge inequivalent vacua of the theory — at least in the YM-coupling regime which we believe to be covered by this picture. Let us remark that all this just follows from the fact that the associated integrable system is obtained by the equations associated to susy preserving BPS field configurations.

Due to the uncontestably partial view offered by perturbative formulation of string theories, during the last years the interest has been extended to the study of its non-perturbative sectors. The concept of strong–weak duality led to a new picture of the string theory moduli space which goes under the name of **M**-theory [7,8]. The spectrum of the $D = 11$ corner of **M**-theory contains membranes and **M**5-branes states whose dynamics has not been fully understood by now. Under compactifications **M**-theory generates, in particular, all **D**-brane states. One might be interested in understanding if the integrable system framework can then be promoted in some way also to the study of **M**-branes states.

A really odd object in **M**-theory is represented by the **M**5-brane since, due to the self-duality of the relevant 2-tensor which lives on it, its relative gauge theory is of a non-Lagrangian type and then much harder to be understood. On top of it, another and harder problem seems to be raised by the lack of a proper formulation in the case of (non-Abelian) higher rank gauge theories which should apply to the multi-**M**5-brane theory.

As far as the single **M**5-brane theory is concerned, the determination of its partition function is available, at least in some specific cases, but its multibrane analogous still resists several attaches.

A simplified configuration, which is more suitable to be studied, is the case in which the **M**5-brane world-volume is of the factorized form $T^2 \times M_4$, where T^2 is a 2-torus and M_4 a four-manifold. In this case [9], the analysis of the CFT living on the **M**5-brane can be attached by reducing to the limit of vanishing 2-torus volume and landing on a four-dimensional gauge theory.

In the case which we will study, the resulting gauge theory is the twisted version of SYM with $\mathcal{N} = 4$ and $U(N)$ gauge group which has been considered in [10]. Let us notice here that once the spectrum of the theory is given, its two derivative low energy effective theory comes out to be automatically topological. This suggests that in such compactifications (see later on for a more precise picture) there is a substantial decoupling of the world-volume theory from the bulk parameters so that in makes sense to consider the N **M**5-branes bound state in isolation. This is possible if in particular the corresponding **M**5-brane anomalies vanish.

On the other hand, the same theory can be studied in an equivalent limit of vanishing M_4 volume thus showing the naturalness of the results in [10], where the structure of the partition function was exhibited as corresponding to a two-dimensional toric model [11].

In this paper, we will try to add a new piece to this brane–gauge theory correspondence by proposing a solution for the twisted $\mathcal{N} = 4\, U(N)$ theory with M_4 a simply connected Kaehler four manifold. Our analysis generalizes the one given in [12] where M_4 was a $K3$ surface, to the case in which M_4 is a simply connected Kaehler manifold with $h^{(2,0)} > 0$. This will be done by exploiting the structure of the gauge inequivalent vacua space of the theory as a space of holomorphic covering of the base four manifold.

This article is organized as follows. In the next section, we will recall how the twisting gets generated in generic CY case and exploit the natural link between twisted Higgs–Hermitian configurations describing **D**-brane bound states and spectral covers. In Section 3, the elliptic genus associated to a single **M**5-branes is discussed in full detail in order to ideologically justify the subsequent developments. In Section 4, the case of N **M**5-branes is solved by the explicit solution of the relevant moduli space identification problem. Conclusions and open questions are contained in Section 5.

2. Refreshing the twist and the spectral covers

Let M_D be a D-dimensional manifold ($D = 10$ for type II theories and $D = 11$ for **M**-theory) and let W_p be a $(p + 1)$-dimensional hypersurface smoothly embedded in M_D with $p + 1 \leq D$.

If W_p is flat, the low energy effective theory of NDp-branes in type II theories whose world-volume coincides with W_p is the $(p+1)$-dimensional reduction of the $\mathcal{N} = 1$ SYM in 10 dimensions. This represents, at weak coupling, the theory of open strings ending on the branes. In particular, the transverse $D - p - 1$ bosonic fields, which are in the adjoint of the gauge bundle and vectors with respect to the transverse unbroken rotational group, are interpreted as representing the transverse motions of the brane itself.

Let us now turn to the more complicate case in which W_p is not a flat manifold. In particular, it means that there is no possible choice for the embedding functions to produce a flat induced metric. It is possible that the normal bundle is nonetheless flat, due to some particular structure of the embedding. In this case, one is led to study SYM theory on the curved W_p and then the analysis carried out in [13] applies. We will concentrate here in cases in which the normal bundle is not flat.

The generic situation arises as follows. Let us consider a given M_D target space preserving some fraction of susy. This implies the existence of covariantly constant spinors on it and hence a holonomy reduced structure in the target space. On the other side, the susy left on the brane sitting on W_p will be determined by the relative orientation of the reduced spin bundle under the reduced holonomy. The supersymmetric cycle condition [14] then relates the normal and the tangent bundles in a non-trivial way. This implies that the spinor bundle $\mathcal{S} = ST(W_p) \otimes SN(W_p)$, to which the spinors belong, admits a representation as a sum of integer spin representations of the tangent bundle itself. This induces then the twisting of the supersymmetric SYM theory which would be present in the flat case. In particular, the superalgebra itself get twisted to the appropriate cohomological algebra. Several examples of the above mechanism can be found both in [2,15], where a general discussion about the induction of the twist by the normal bundle is presented.

Let us concentrate on (partial) wrapping along even susy Kaehler cycles. Notice that, while for a single brane the BPS stable states are specified by particular holomorphic embeddings in the target space, in the N branes case the BPS stable states are determined by an integrable system. In several cases the normal bundle splits as $N(W_p) = \mathcal{N}(W_p) \oplus \bar{\mathcal{N}}(W_p)$ and the integrable system is built from a holomorphic vector bundle E and a holomorphic section $\Phi^{\mathcal{N}}$ of $\mathrm{Adj}(E) \otimes \mathcal{N}(W_p)$ as

$$\mathcal{D}\Phi^{\mathcal{N}} = 0, \qquad F^{(2,0)} = 0, \qquad \omega \cdot F + [\Phi^{\mathcal{N}}, (\Phi^{\mathcal{N}})^{\dagger}] = 0,$$

where ω is the Kaehler form on the cycle.

In particular, then the spectral cover for the system

$$\det(\Phi^{\mathcal{N}} - \phi \mathbf{1}_N) = 0$$

materializes in the total space of the holomorphic factor of the normal bundle and classify its solutions. Let us here stress the obvious fact that the normal bundle total space is nothing but (a part of) the target space itself. Hence this curve is identified with the relevant wrapped world-volume of the N branes system and is the analogous of the SW curve in the curved case.

The case of the **M**5-brane is technically more complicated and much less understood, but we accept it to follow a similar pattern. The first problem is that the world-volume theory is not an SYM theory, but a gauge theory of a self-dual tensor multiplet. Secondly, the higher rank case is not either formulated. In this case, one can try to understand it as the strong coupling limit of the **D**4-brane theory, but the very structure of the **M**-interaction [16] remains unknown by now. A possibility to investigate the **M**5-brane theory structure is anyhow to wrap it on a product cycle in such a way to produce a twisted supersymmetric YM theory, which is then topological, on a factor. Assuming it not depending on the ratio of the volumes of the factor cycles, it is then possible to try to gain some informations just by rescaling inversely the volumes and see if the theory happens to have a simpler picture. Notice that so doing we are pretending the world-volume theory of the **M**5-brane to be effectively decoupled with respect to the bulk fields. On the other hand, it is well known that the theory of the **M**5-brane suffers from inborn and inflow anomalies and that therefore, to have a consistent substantially decoupled brane configuration one has to verify that both these contributions to the anomaly cancel separately. Of course, the coherence of the full out-come result is a test for the **M**-theory conjecture.

3. The single M5-brane partition function

The geometric set-up that we refer to is the following [30]. We consider **M**-theory on $W = Y_6 \times T^2 \times R^3$, where Y_6 is a Calabi–Yau threefold of general holonomy. Let M_4 be a supersymmetric simply connected four-cycle in Y_6 which we take to be a representative of a very ample divisor [30]. Notice that M_4 is automatically equipped with a Kaehler form ω induced from Y_6. We consider then one 5-brane wrapped around $C = T^2 \times M_4$.

The bosonic spectrum of the world-volume theory of this 5-brane is given by a 2-form V with self-dual curvature and five real bosons taking values in the normal bundle N_C induced by the structure of the embedding as $T_W|_C = T_C \oplus N_C$. Passing to the holomorphic part and to the determinants and using the properties of Y_6, it follows that the five transverse bosons are, respectively, three non-compact real scalars ϕ_i and one complex section of $K_{M_4} = \Lambda^{-2}T^{(1,0)}_{M_4}$, the canonical line bundle of M_4, Φ. The (partially) twisted chiral (0, 2) supersymmetry completes the spectrum.

Notice that to have a self-consistent theory of a 5-brane in isolation, it has to be anomaly free by itself and also from the inflow point of view. These conditions are fulfilled if [21]

$$(p_1(T_C) - p_1(N_C))^2 = 4p_2(T_C), \qquad p_2(N_C) = 0.$$

In our case, we have

$$p(T_C) = c(T^{(1,0)}_{M_4})c(T^{(0,1)}_{M_4}) = 1 + (2c_2 - c_1^2) + c_2^2,$$
$$p(N_C) = c(K_{M_4})c(\bar{K}_{M_4}) = 1 - c_1^2,$$

where $c_i = c_i(T^{(1,0)}_{M_4})$, and the above conditions are identically satisfied. Let us notice here that this requirement is enough also for the cancellation of the anomalies in the case of several 5-branes, at least as it is given in [22].

Even if a precise recipe to give a Lagrangian formulation of the theory is not known, fortunately we have the possibility to calculate the partition function of the 5-brane in the limits in which the volume of one of the two factors of $C = T^2 \times M_4$ vanishes. If the two results will agree we can believe that really the theory depends on the product of the two volumes only and then promote this feature to the $N > 1$ case too.

Let us reduce to zero the volume of M_4. In this case [11,20], V gives a vector a, b_+ left and b_- right real compact scalars, Φ gives $b^{(0,2)}$ complex scalars and ϕ_i three real scalars. All in all we have $b_2 + 2$ left and $4b^{(0,2)} + 4$ right scalar bosons. The chiral fermions will provide correspondingly $4b^{(0,2)} + 4$ right periodic fermions and no left fermions.

The elliptic genus is defined as

$$\mathcal{E} = \frac{(\operatorname{Im}\tau)^{d/2}}{V_d}\mathrm{Tr}_{RR}[(-1)^F F_R^{\sigma/2} q^{L_0}\bar{q}^{\bar{L}_0}],$$

where d is the number of non-compact scalar bosons, V_d their zero-mode volume and $(0, \sigma)$ are the supersymmetries of the model. By general arguments, $\mathcal{E}$ is a $(-\frac{1}{2}d, -\frac{1}{2}d) + (0, \frac{1}{2}\sigma)$ modular form.

In our case, we have $\sigma = 2b_+ + 2 = 4b^{(2,0)} + 4$ right fermions and $d = 3 + 2b^{(2,0)} = 2 + b_+$ non-compact scalar bosons by the dimensional reduction giving the elliptic genus to be a modular form of total weight $(-1 - \frac{1}{2}b_+, \frac{1}{2}b_+)$.

We can calculate the elliptic genus using standard techniques in CFT_2 as [1]

$$Z_1^M = \left(\frac{\theta_\Lambda(q,\bar{q})}{\eta^{b_-}(q)\eta^{b_+}(\bar{q})}\right) \cdot \left(\frac{1}{\eta(q)\eta(\bar{q})}\right)^{3+2b_{(2,0)}} \cdot (\eta(\bar{q}))^{4b_{(2,0)}+4} = \frac{\theta_\Lambda(q,\bar{q})}{\eta^{\chi}(q)},$$

where the first factor comes from the compact bosons, the second from the non-compact

[1] We assumed $b_1 = 0$ because of the ampleness of M_4 as a divisor in Y_6. If this would not be the case, we then had a further multiplicative factor $(\eta/\bar{\eta})^{2b_1}$.

bosons and the last one comes from the fermions. The function

$$\theta_\Lambda(q,\bar q)=\sum_{m\in\Lambda} q^{1/4(m,*m-m)}\bar q^{1/4(m,*m+m)}$$

is a generalized θ-function, Λ is the lattice of integer period in $H^2(M,R)$, $(\cdot,\cdot)$ is the intersection form, $*m$ is the Hodge dual of m and $q=\mathrm{e}^{2\pi i\tau}$. The unmatched left scalar bosons produce the overall factor of $\eta(q)^{-\chi}$, where $\eta(q)=\sum_n(-1)^n q^{3/2(n-(1/6))^2}$ is the Dedekind η-function and $\chi=2+b_2$ is the Euler characteristic of M. In particular, Z_1^M is a modular form of weight $(\frac{1}{2}b_-,\frac{1}{2}b_+)+(-\frac{1}{2}\chi,0)=(-1-\frac{1}{2}b_+,\frac{1}{2}b_+)$ as it had to be.

Let us now reduce to zero the volume of the torus. The six-dimensional V field reduces then to a vector A and to a real scalar b, while Φ just generates a section of the canonical bundle B and the three scalars ϕ_i remain three scalars. The twisted $U(1)$ theory spectrum is given by the gauge field A, a self-dual 2-form $B^+=B+\omega b+\bar B$ and three scalars, as far as the bosonic part is concerned, and by its twisted-susy counterpart given by Grassmann valued two self-dual 2-forms, two 1-forms and two scalars.

Up to the gauge field A, which is a connection for the gauge bundle all the other fields are also sections of the adjoint bundle, and therefore in the single brane case, uncharged.

As it is naturally expected, in the calculation of the $U(1)$ partition function, once also the ghosts are properly taken into account, there is a complete cancellation between all the fluctuation contributions and we are left with an exact expression of the partition function coming only from the zero-mode contributions.[2] Therefore, the contribution from the gauge sector is the classical one coming from the $U(1)$ gauge field which can be evaluated to be [17]

$$q^{-\chi/24}\theta_\Lambda,$$

where $\theta_\Lambda(q,\bar q)=\sum_{m\in\Lambda}q^{1/4(m,*m-m)}\bar q^{1/4(m,*m+m)}$ is exactly the same object that we have found before [11].

To obtain the full partition function one has to add up also the contribution coming from point-like degenerate instantons which amounts to a multiplicative factor of

$$\sum_n q^n \dim H^*\left(\frac{M_4^n}{S_n}\right)=\frac{1}{\prod_{n>0}(1-q^n)^{2+b_2}}=q^{\chi/24}\eta^{-\chi}.$$

The nature of this factor can be traced back to [31].

Therefore, all in all, we obtain

$$Z_1^M=\frac{\theta_\Lambda}{\eta^\chi},$$

which is the same result that we had in the two-dimensional computation.

[2] In detail, denoting by Δ_i the determinants of the Laplacian on i-forms, the bosonic sector contributes with $(\Delta_0/\sqrt{\Delta_1})\cdot(1/(\Delta_2^+)^{1/2})\cdot(1/(\Delta_0)^{3/2})$, while the fermionic sector contributes as $[(\Delta_2^+)^2\cdot(\Delta_1)^2(\Delta_0)^2]^{1/4}$ giving all in all 1. Also, an exact cancellation between the zero modes volumes takes place and no $\mathrm{Im}\,\tau$ factors are present because of the topological symmetry of the twisted theory.

4. The N M5-branes partition function

The N 5-brane theory is not known. What we assume to be true — and will show to be consistent — is the following.

By shrinking the M_4 volume to zero we find a toroidal σ-model valued in the moduli space $\mathcal{M}$ of BPS-like field configurations of the twisted $\mathcal{N}=4$ SYM $U(N)$ theory on M_4 itself, in the spirit of Bershadsky et al. [18]. By shrinking the torus volume to zero we find directly the twisted $\mathcal{N}=4$ SYM $U(N)$ theory on M_4.

The torus contribution will amount to the elliptic genus of the (stratified) moduli space $\mathcal{M}$, while the same object is to be obtained as the M_4 partition function (with fermionic insertions). This, as explained in [10], counts the Euler characteristic of $\mathcal{M}$ and fits exactly this identification [9].

Notice that in the limit of zero volume of the torus, **M**-theory on $T^2 \times Y_6 \times R^3$ is dual to perturbative type IIB on $R \times Y_6 \times R^3$ with coupling τ and that the N **M**5-brane system gets mapped to a bound state of N **D**3-branes wrapped around M_4. Their low energy description is in fact given by the twisted $\mathcal{N}=4$ SYM with gauge group $U(N)$ as given above.

4.1. *The moduli space and the spectral cover*

In this section, we calculate the relevant moduli space of the twisted $\mathcal{N}=4$ Yang–Mills theory.

The twisted $U(N)$ theory spectrum is given by the gauge field A, a self-dual 2-form B^+ and three scalars, as far as the bosonic part is concerned, and by its twisted-susy counterpart given by Grassmann valued two 2-forms, two 1-forms and two scalars. Beside the gauge field A, which is a connection for the gauge bundle E, all other fields are also sections of the adjoint bundle $\mathrm{Adj}(E)$.

Let us start from the susy fixed point condition for the twisted $\mathcal{N}=4$ theory with gauge group $U(N)$. These are given by the following equations [10]:

$$\omega \wedge F + [B, \bar{B}] = 0, \qquad F^{(2,0)} = 0, \qquad \bar{D}B = 0, \tag{4.1}$$

where ω is the Kaehler form on M, F the curvature of the relevant gauge bundle E, B is a section of $\mathrm{End}(E) \otimes K$ with K the canonical line bundle of M while $\bar{B}$ is its conjugate.

We associate to each solution of (4.1) a spectral four manifold defined by the equation

$$\det(k 1_N - B) = 0, \tag{4.2}$$

where k is an indeterminate taking values in the total space of the canonical line bundle K, where Eq. (4.2) defines a rank N holomorphic covering of M. Let us call Σ this resulting Kaehler manifold.

To construct the solution explicitly, start writing the spectral surface equation as

$$0 = \det(k 1_N - B) = k^N + \sum_{i=0}^{N-1} k^i \beta_i, \tag{4.3}$$

where β_i are holomorphic sections of K^{N-i} which can be explicitly be given by symmetric polynomials in B.

Let us first consider the case in which Σ is connected. This means that the polynomial in (4.3) is irreducible. By using the complexification of the gauge group, we can reduce B to its bare bones, i.e. write it as

$$B = \begin{pmatrix} -\beta_{N-1} & -\beta_{N-2} & \cdots & \cdots & -\beta_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{4.4}$$

while the reduced complex bundle remains

$$E = E_N = \mathop{\oplus}_{i=1}^{N} L^{N+1-2i}, \tag{4.5}$$

where L is a line bundle over M such that $L^2 = K$. This data reduction from principal Higgs bundles spectral surface data is called Abelianization in the mathematical literature (see, for example [23]). Notice that only if N is even K has really to admit a square root L. Moreover, the overall phase is fixed by the $SL(N, C)$ complexified structure group. As we will explain later on, it is not at all a coincidence that E_N in (4.5) is modeled on the N-dimensional irreducible $SU(2)$ representation.

In general, the B characteristic polynomial splits in irreducible monic factors [3] of degree $a > 0$ as

$$\det(k1_N - B) = \prod_a [P_a(k)]^{n_a}, \tag{4.6}$$

where $N = \sum_a a \cdot n_a$. The relative covering spectral surface factors then in connected components as $\Sigma = \bigcup_a [\Sigma_a]^{n_a}$, where Σ_a is the surface given by the equation $P_a = 0$ and is given by a specific choice of L.

Let us notice that the multiplicities n_a indicates the presence of a non-trivial unbroken subgroup $J = S_{n_1} \times (S_{n_2} \bowtie \mathbf{Z}_2) \times (S_{n_3} \bowtie \mathbf{Z}_3), \ldots$ of the $U(N)$ Weil group S_N which has to be factorized out in order to get the true moduli space. This determines the moduli space of solutions of (4.1) to be the Hilbert scheme of holomorphic coverings of M_4 in Y_6.

The factorization (4.6) induces the analogous structure on the vector bundle as $E = \oplus_a E_a^{\otimes n_a}$ which is in fact modeled on the generic reducible N-dimensional representation R_N of $SU(2)$. In terms of irreducible ones we have in fact $R_N = \sum_a [R_a^{(\mathrm{irr})}]^{n_a}$, where $R_a^{(\mathrm{irr})}$ is the unique a-dimensional $SU(2)$-representation. The sum over these $SU(2)$ embeddings, which will not be reviewed here, is the sum over the sectors explained at length in [10] (Section 5) and in [12]. Let us point out here that this analysis [24] of the vacua structure fails exactly at the canonical class, which for a generic Kaehler manifold is non-empty, giving rise to locally enhanced gauge symmetry. This corresponds exactly in the spectral cover picture to the branching locus. Notice also the natural action of the J subgroup of

[3] For example, let us consider the case $N = 2$ and the polynomial $k^2 + b_0$, where b_0 is a holomorphic section of K^2. We have factorization if we can write $b_0 = s^2$ with s a holomorphic section of K.

the Weil group on R_N which symmetries with respect to the multiplicities of the $SU(2)$ irreps. Anyhow, this field theory analysis is valid only if $b_+^{M_4} > 1$ when the moduli space of holomorphic deformations of M_4 in Y_6 is of positive dimension.

Generically, the canonical divisor $K = \sum_l C_l$ can be considered to be a sum of irreducible curves C_l. The branching locus of the covering is the zero locus of the discriminant of the B characteristic polynomial (4.2) which is a holomorphic section of $K^{N(N-1)}$. Therefore the branching locus can be represented as a collection of the irreducible components of the canonical divisor itself as $H = \bigcup_l{}' C_l'$ for a subset $\{l'\} \in \{l\}$. Unfortunately, we cannot still conclude about the structure of the spectral cover canonical class, K_Σ because of a residual set in the lifting of the branching locus. On the other hand, because of ampleness, we have in any case that K_Σ is still very ample by construction and therefore its Betti numbers are computable directly by making use of the formulas obtained in [30] as

$$b_+(\Sigma) = -1 + \int_Y \frac{1}{3}\Sigma^3 + \frac{1}{6}\Sigma c_2(Y), \qquad b_-(\Sigma) = -1 + \int_Y \frac{2}{3}\Sigma^3 + \frac{5}{6}\Sigma c_2(Y),$$

and the fact that $\Sigma = M^N/H$. Specifying Y, M and H one can in principle explicitly calculate also the intersection form on Σ.

4.2. $K3$ as the totally reducible case and the T^2 multiplicity

Let us now go back to the 5-brane point of view: the 5-brane is taken to be N times wrapped around $W = T^2 \times M_4$.

In [2] it has been shown that if M_4 is a $K3$, then the partition function of N coinciding branes on it is obtained by summing over all the possible rank N holomorphic coverings of the T^2 over itself.

If M is a $K3$ surface, then, since $K3$ does not admit any non-trivial holomorphic line bundle for a generic complex structure, there is no non-trivial solution of (4.1). Therefore, in calculating the N 5-branes partition function, the sum over all the possible holomorphic coverings of C over itself reduces to the T^2 sector since the only irreducible holomorphic covering of $K3$ results to be the trivial one. The counting is then made following [26] by summing over all the possible holomorphic coverings of T^2 over itself, which is by summing over all the resulting values of the covering tori moduli, the result being given by the Hecke transform of order N of the single 5-brane partition function on W:

$$Z_N^{K3} = \mathcal{H}_N Z_1^{K3}. \tag{4.7}$$

Let us comment about the important fact that the **M**5-brane point of view tells us the exact way we have to count the multiplicity of the trivial covering, i.e. it is the **M**5-brane point of view which fixes the form of the Hecke transform induced by the symmetric space elliptic genus formula.

Our aim is now to generalize the above result to a more general case. The counting of BPS bound states of N **M**5-branes is then the counting of holomorphic self-covering of C on itself with total rank N. They can be classified in terms of partitions of N in positive integers as $N = \sum_a a \cdot n_a$ such that each term in the sum represents a connected component of the covering and each component is a connected n_a-folded holomorphic covering of the torus times a connected a-folded holomorphic covering of M_4.

Let us now associate to each connected component a partition function Z_{a,n_a}. Then, for each given partition $N = \sum_a a \cdot n_a$, the total contribution to the partition function will be given by the product

$$\prod_a Z_{a,n_a}.$$

From the $K3$ case analysis we learn that the T^2 multiplicity n_a is naturally kept into account by the Hecke operator as $Z_{a,n_a} = H_{n_a} Z_{a,1}$, where H_n is the Hecke transform of order n. The Hecke operator structure can be recognized from the field theory side as being the result of the process of redefinition of the coupling of the gauge theory due to the rescaling of the traces weights [2] and we have shown it to correspond just to the J symmetrization operation. On the other side, its appearance from the torus point of view is naturally obtained as the result of the relevant instanton sum in the σ-model [19]. Let us anticipate here that the contribution $Z_{a,1}$ will turn out to be a modular form of weight $(-1-\frac{1}{2}b_+^{\Sigma_a}, \frac{1}{2}b_+^{\Sigma_a})$. Since the Hecke transform preserves modularity, our final result admits a natural interpretation from the two-dimensional point of view as the elliptic genus of a specific sigma model valued in the co-homology of the moduli space of vacua that we just calculated. The calculation is much similar to the one that we already gave for the rank one case.

We are then left with the calculation of the four-dimensional field theory contribution in the irreducible case without torus multiplicity.

4.3. The field theory counting

By the topological nature of the twisted $\mathcal{N} = 4$ SYM, our partition functions localize around the solutions of (4.1). Therefore, if one is going to calculate the partition function of N coinciding 5-branes wrapped around C as the partition function of a single 5-brane wrapped N times around C by counting all the possible rank N holomorphic coverings of C on itself, then the relative spectral surface should be kept into account in the covering counting, if solutions with non-trivial B of (4.1) exist.

Let us now proceed by making use of the analysis of the vacua structure of the theory that we just performed. As we have shown above all the leftover calculation is reduced to the irreducible coverings.

The field theory path integral for the twisted susy theory can be performed by semi-classical approximation which, due to the topological features of the twisted theory, turns out to be exact. This can be done just following the analysis in [27], which is given for the untwisted case, adapting it to the twisted $U(N)$ theory as follows. Each generic point in the (irreducible part of the) moduli space $\mathcal{M}_{\mathrm{irr}}$ breaks completely the gauge invariance from $U(N)$ to the maximal compact Cartan torus $U(1)^N$, up to the local enhancing at the canonical class, by singling out a canonical choice for a Cartan subalgebra. Integrating out the non-Cartan valued fields, the semi-classical evaluation of the path integral reduces to a collection of N-Abelian twisted multiplets, one for each sheet of the spectral cover. The lifting technique, which is just the field theory counterpart of the Abelianization for sections of principal Higgs bundles, then gives as an outcome the twisted super-Maxwell theory on the spectral cover

Σ, which is the single 5-brane contribution Z_1^{Σ} that we just had in the previous section, but evaluated on the spectral cover. [4]

The total contribution to the irreducible sector will be then given after a summation over all the possible L line bundles is also kept into account. Assuming $K^{1/2}$ to exist on M_4, we have $L = K^{1/2} \otimes \mathcal{O}_\varepsilon$. Here, $\mathcal{O}_\varepsilon = \sum_l \epsilon_l [C_l]$, where $[C_l]$ are the reductions modulo 2 of (the Poincare dual of) the curves C_l that we introduced before, the label ε indicates the collections of integer numbers $\{\epsilon_l\}$ which satisfy $\epsilon_l^2 = \epsilon_l$ for each l. Recalling that the reduced form of the vector bundle is $E_N = \sum_{i=1}^N L^{N+1-2i}$, we obtain $E_N = \mathcal{O}_\varepsilon^{\otimes N+1} \otimes \sum_{i=1}^N (K^{1/2})^{N+1-2i}$. With this parameterization of the line bundle L, we can write our final result as

$$Z_{a,n_a} = \mathcal{H}_{n_a} Z_{a,1}, \qquad Z_{a,1} = \sum_{\varepsilon} \frac{\theta_{\Lambda^{\Sigma_a}+x}}{\eta^{\chi_{\Sigma_a}}},$$

where $x = [\mathcal{O}_\varepsilon^{\otimes a+1}]$ shifts correspondingly the lattice of integer periods Λ^{Σ_a} on $H^2(\Sigma_a, R)$. Notice that this shift is effective only if a is even.

Needless to say, $Z_{a,1}$ is non-vanishing effectively only if there exists irreducible solutions of the spectral equation of the corresponding orders.

A closer inspection should reveal the correspondence between the sum over the x-classes, once the proper $U(1)$ components are divided out from our formula, and the sum over the SW classes as obtained in [25] for the $SU(2)$ case. Moreover, similar results to the $K3$ case should hold more generally for manifolds M and given values of N such that M does not admit irreducible self-holomorphic coverings of ranks corresponding to partitions of the actual N, beside the one corresponding to $a = 1$ and $n_1 = N$.

Since the partition function Z_1^{Σ} is completely determined by the intersection form on $H^2(\Sigma)$, we could calculate it in full explicit form case by case by specifying the particular cycle in a given Calabi–Yau.

5. Conclusions and open problems

In this note, we used the geometrical properties of the **M**5-brane in **M**-theory to address the problem of calculate the partition function in the twisted $\mathcal{N} = 4$, $D = 4$ SYM on a Kaehler manifold, pushing on a strong geometric interpretation of the first.

Our calculational scheme is quite general and could be tested in different specific configurations. In particular, we think it would be important to understand weather it extends to the $b_+^M = 1$ cases where the absolute rigidity of the 4-cycle causes well-known problems in the evaluation of the partition function, the quantum field theory vacua analysis remaining empty.

It would be very interesting to understand from this general point of view also the problem of holomorphic anomalies. This question is strictly linked to the degeneration corners of the moduli spaces in the reducible locus and can be understood from the spectral cover point of view almost easily in terms of subloci in which the factorization in different connected components of the generically connected spectral surface takes place. This automatically

[4] The lifting technique has been originally performed in the two-dimensional case corresponding to Matrix String Theory in [28,29].

generates extra zero-modes representing other **M**5-branes bound states at threshold causing the holomorphic anomaly. The technical and less trivial part is then the full identification of that contribution in the partition function.

The most severe bound in performing the above programs is that admittedly our formula is rather implicit because of the lack of the necessary mathematical technology to calculate in the generic case the canonical class of the spectral surface and its intersection form.

Another interesting possible development could be to try to extend this analysis to **M**-theory on $R^3 \times Y_8$, with Y_8 an elliptically fibered CY fourfold and the **M**5-branes wrapped around an elliptically fibered supersymmetric 6-cycle in it in order to try to extract more informations about the structure of the N **M**5-brane world-volume theory.

With the same attitude, a possible very interesting development would be to generalize and clarify the genus expansion given for the $SU(2)$ case in [25] to the $U(N)$ case obtained here. Doing this, one could be able to exhibit the form of the N **M**5-brane partition function as a (tensionless) string theory one.

As a possible application one could use the results obtained here to **M**-theory black hole entropy counting generalizing the analysis in [30,32]. The result of our investigation suggests the existence of several contributions relative to the different possible holomorphic covering of $C = T^2 \times M_4$ in the form of a fine structure in the entropy formula. This seems to lead to subleading **M**-theory corrections to the full 11-dimensional supergravity result as discussed in [33–35]. Anyhow, all the sectors counting, done as

$$(\Delta S)_{a,n_a} = 2\pi\sqrt{\frac{n_a c_L^{\Sigma_a}}{6}},$$

where $c_L^{\Sigma_a} = \int_{Y_6}(\Sigma_a^3 + \Sigma_a \cdot c_2(Y_6))$, will yield in the large cycle volume and large N limit, the appropriate result.

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